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# A complete Crocco integral for two-dimensional laminar boundary layer flow over an adiabatic wall for Prandtl numbers near unity

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The so-called Crocco integral establishes a relation between the velocity and temperature distributions in steady boundary layer flow. It corresponds to an exact solution of the flow equations in the case of unity Prandtl number and an adiabatic wall, where it reduces to the condition that the total enthalpy remains constant throughout the boundary layer, irrespective of pressure gradient and compressibility. The effect of Prandtl number is usually incorporated by assuming a constant recovery factor across the entire boundary layer. Strictly, however, this modification is in conflict with the conservation-of-energy principle. In search of a more complete expression for the Crocco integral the present study applies an asymptotic solution approach to the energy equation in constant-property flow. The analysis of self-similar boundary layer solutions results in a formulation of the Crocco integral which correctly incorporates the effect of Prandtl number to first order, and that is complete in the sense that it satisfies the energy conservation requirement. Furthermore, the result is found to be applicable not only to self-similar boundary layers, but also to provide a solution to the laminar flow equations in general as well. The effect of varying properties is considered with regard to the extension of the expression to more general flow conditions. In addition to the asymptotic expression for the Crocco integral, asymptotic solutions are also obtained for the recovery factor for various classes of flows.

## 1. Introduction

Owing to the similarity in the phenomena that provide the transport of momentum and heat in a boundary layer flow, a close link exists between the flow velocity and the local enthalpy in the flow. This is expressed by the so-called Crocco(–Busemann) relation or the Crocco integral, in acknowledgement of the pioneering work which L. Crocco (1932) performed in the field of thermal effects in boundary layer flow. Such a relationship has great practical significance, both for experimental purposes where it can provide temperature information where such is not explicitly available (Crabtree 1954; Fernholz & Finley 1980; Motallebi 1994), as well as in numerical studies in which the temperature can be obtained from an algebraic relation rather than having to solve the energy transport equation (Lindhout *et al.* 1981; Kiss & Schetz 1993).

Following the original work by Crocco, the subject of heat transfer in laminar boundary layer flows has received much attention. In this respect it is important to refer to the large body of analytical studies that appeared in the 1950s and

1960s. In particular the treatment of the Falkner–Skan boundary layers in constant-property flow and the flat-plate boundary layer in compressible flow, where similarity is obtained in both the velocity and temperature distributions, has laid the theoretical basis for the understanding of aspects such as heat transfer and enthalpy recovery, and the effect of pressure gradient, Prandtl number and compressibility. Good overviews of the major findings of these studies have been given by Stewartson (1964), Schlichting (1979), Anderson (1989), White (1991) and Schlichting & Gersten (1997), whereas Fernholz & Finley (1980) and Dussauge *et al.* (1996) have given a renewed account of the Crocco relation in connection with compressible turbulent flows.

The derivation of the original Crocco integral relation can be found in standard textbooks, such as the sources mentioned above. Basically, for steady two-dimensional boundary layer flow over a stationary and impermeable wall it holds that at unity Prandtl number $Pr$ and either zero pressure gradient or adiabatic wall conditions, the temperature (throughout this study described in terms of the enthalpy $h$) can be expressed as a function of the main velocity component $u$, for any arbitrary viscosity function $\mu(T)$:

$$h = h_e + (h_{aw} - h_e)\left(1 - \frac{u^2}{u_e^2}\right) + (h_w - h_{aw})\left(1 - \frac{u}{u_e}\right) \tag{1.1}$$

where the subscripts $w$ and $e$ refer to conditions at the wall surface and in the (adiabatic) external stream, respectively; $h_{aw}$ is the enthalpy corresponding to the adiabatic wall temperature:

$$h_{aw} = h_e + r\tfrac{1}{2}u_e^2. \tag{1.2}$$

The above expression defines the recovery factor $r$, which is equal to 1 in the case of unity Prandtl number. Restricting the further discussion in this paper to the case of zero heat transfer, the Crocco integral then reduces for $Pr = 1$ to the condition that the total enthalpy $H = h + \frac{1}{2}u^2$ remains constant over the entire boundary layer thickness.

Based on the work of Van Driest (1952) and Walz (1966), who extended Crocco's analysis of the flat plate to arbitrary Prandtl number, the effect of $Pr$ is commonly incorporated by applying a direct generalization of (1.1). This corresponds to the assumption that a constant recovery factor can be applied throughout the entire boundary layer, also in the case where $r$ differs from 1, so that for an adiabatic wall:

$$h = h_e + r\tfrac{1}{2}(u_e^2 - u^2) \tag{1.3}$$

which implies:

$$H = H_e + (r-1)\tfrac{1}{2}(u_e^2 - u^2). \tag{1.4}$$

The value of $r$ is based on the temperature recovery at the wall, and derived from theoretical studies of self-similar boundary layers in constant-property flow to be approximately $r = Pr^{1/2}$ for laminar flow, irrespective of pressure gradient (Pohlhausen 1921; Tifford & Chu 1952; Brun 1956; Le Fur 1960; Schlichting 1979), which was confirmed analytically and experimentally to be a viable approximation for compressible flow as well (Kaye 1954; Van Driest 1959).

A serious objection to this modification of the Crocco integral, however, is that the resulting enthalpy distribution violates the conservation-of-energy principle. This is reflected by the integral energy balance which for a steady two-dimensional flow with constant $H_e$ over a stationary, impermeable wall is

$$\frac{\partial}{\partial x}\int_0^{\delta} \rho u(H - H_e)\mathrm{d}y = q_w(x), \tag{1.5}$$

where $x$ is measured along the surface in the direction of the external stream and $y$ is the distance to the wall. In the absence of surface heat transfer ($q_w = 0$) this reduces to:

$$\int_0^{\delta} \rho u(H - H_e)\mathrm{d}y = 0. \tag{1.6}$$

This local integral energy condition is clearly not satisfied in general by equation (1.4), apart from the trivial case when $r = 1$. The reduction of the total enthalpy near the wall due to the incomplete heat recovery (for realistic gases where $Pr < 1$), is balanced by a flow region near the outer flow where $H > H_e$, an effect which is not reproduced by an expression of the type (1.3).

Although this defect has been reported already by e.g. Schubauer & Chen (1959), the author knows of no attempt to modify the Crocco integral accordingly, at least not in a systematic and rigorous manner (cf. also Fernholz & Finley 1980). Schubauer & Chen (1959) proposed the use of a variable recovery factor; however, they did not suggest how it could be implemented. This approach would involve unrealistic recovery values occurring in the outer region of the boundary layer, which does not provide a correct description of the underlying physical mechanism: the overshoot in the total temperature is not due to a variation of the heat recovery, but is caused by additional heat transport in the flow, as will be shown in §3.1. Hence, the highly empirical and *ad hoc* nature of such a modification would probably make it of only very limited use.

For the case of turbulent flat-plate flow, Whitfield & High (1977) derived a modified velocity–temperature relation for non-unity Prandtl numbers, allowing the description of total temperature overshoot. It was obtained from the zeroth- and first-order perturbation solution, for $Pr$ near 1, of the Crocco-transformed energy equation (i.e. with $u$ as independent variable), by prescribing the relation between the shear stress and velocity profile. This allowed them to succesfully reproduce experimental total temperature data, but the validity of the expression is evidently limited to a single, specific class of flows, namely that of (turbulent) zero-pressure-gradient flows in near equilibrium.

The purpose of the present study is to derive a general, modified expression for the (laminar) Crocco integral under the condition that $Pr$ differs from 1, that is in correspondence with physical conservation principles. Similar to the approach followed by Whitfield & High (1977), an asymptotic analysis is performed for $Pr$ near 1, but here it is applied directly to the complete laminar flow equations, and without making any additional assumptions about the flow field.

First, consideration is given to self-similar solutions of the constant-property two-dimensional boundary layer equations. Next, the extension to more general conditions is addressed. As a side product of this investigation of the Crocco integral, asymptotic expressions are also obtained for the Prandtl-number dependence of the recovery factor for various classes of flows.

## 2. Self-similar boundary layers in constant-property flow

### 2.1. *Governing equations*

The momentum and energy equations governing the steady two-dimensional laminar boundary layer flow with respect to the Cartesian coordinate frame $(x, y)$, with $(u, v)$ the corresponding velocity components and $p$ the pressure, are given by

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y}, \tag{2.1}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\mathrm{d}p}{\mathrm{d}x} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial}{\partial y} \frac{\mu}{Pr} \frac{\partial h}{\partial y}. \tag{2.2}$$

An adiabatic external flow is assumed so that $\mathrm{d}p/\mathrm{d}x = -\rho_e u_e \mathrm{d}u_e/\mathrm{d}x = \rho_e \mathrm{d}h_e/\mathrm{d}x$. Throughout the present study constant specific heat and Prandtl number are assumed. If in addition the density $\rho$ and viscosity $\mu$ are constant, self-similar solutions can be obtained, where the non-dimensional velocity and enthalpy distributions are functions only of the scaled transverse coordinate $\eta$:

$$f'(\eta) = \frac{u}{u_e}, \qquad \theta(\eta) = \frac{h - h_e}{\frac{1}{2} u_e^2}, \qquad \eta = y \left( \frac{m+1}{2} \frac{\rho u_e}{\mu x} \right)^{1/2}. \tag{2.3}$$

The requirement for similarity is a constant value of the pressure-gradient parameter $m = (x/u_e)\mathrm{d}u_e/\mathrm{d}x$, implying that the external stream velocity varies as $u_e \sim x^m$ (wedge flow). This transformation reduces the governing equations to a set of ordinary differential equations (see e.g. Schlichting 1979):

$$f''' + f f'' + \beta(1 - f'^2) = 0, \tag{2.4}$$

$$\theta'' + Pr f \theta' - 2Pr \beta f' \theta = -2Pr f''^2, \tag{2.5}$$

where $\beta = 2m/(m+1)$ and with boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$, $\theta(\infty) = 0$ and $\theta'(0) = 0$ for a thermally isolated wall. Owing to the assumption of constant-property flow the $f$-equation is decoupled from the the energy equation, whereas the latter is linear in $\theta$.

The solution for $Pr = 1$ is given by $\theta = 1 - f'^2$, which corresponds directly to (1.3) with $r = 1$. As discovered by Pohlhausen (1921), the solution for $\theta$ with arbitrary $Pr$ can be formulated analytically in the case of the flat plate ($\beta = 0$), with numerical evaluation yielding that the effect of $Pr$ on the recovery factor can be approximated as $r = Pr^{1/2}$. The second-order term in the asymptotic expansion of the dependence of $r$ on $Pr$ was considered by Spence (1960).

Numerical solutions investigating the effect of $Pr$ and $\beta$ indicated the square-root dependence of $r$ on $Pr$ to be approximately independent of pressure gradient (Tifford & Chu 1952; Brun 1956). This was supported by the analysis of Le Fur (1960), who obtained an approximate expression for the recovery factor that is independent of pressure gradient, from combining first-order subsequent-approximation solutions to both the velocity and the temperature equation.

Further results of numerical solutions for several values of $Pr$ and $\beta$ can be found in for example Gersten & Körner (1968). These also address the effect of a non-zero normal velocity at the wall, which is excluded in the present discussion, revealing among other things how with increasing suction the recovery factor tends towards unity, irrespective of pressure gradient.

In addition, Herwig (1987) investigated the effect of compressibility and heat transfer on the wedge-flow solutions with varying properties, by deriving sensitivity factors for various temperature dependencies from a first-order perturbation of the constant-property solution, again for given Prandtl number $Pr$ and wedge parameter $m$ values (see also Schlichting & Gersten 1997).

In the following subsections an asymptotic solution approach is applied to the similarity equations, in order to investigate in more detail the effect of $Pr$ on the solution, with particular emphasis placed on the relation between the velocity and enthalpy solutions.

### 2.2. *Asymptotic solution approach*

To study the effect of Prandtl number on the solution of the energy equation for $Pr$ different from unity, the enthalpy is expresed as an asymptotic series:

$$\theta(\eta) = \theta_0(\eta) + \epsilon\theta_1(\eta) + \epsilon^2\theta_2(\eta) + ... \tag{2.6}$$

with respect to the perturbation parameter $\epsilon = Pr - 1$. Substitution of the series into (2.5) and collecting equal powers of $\epsilon$, yields the following set of equations:

$$\theta_0'' + f\theta_0' - 2\beta f'\theta_0 = -2f''^2, \tag{2.7}$$

$$\theta_1'' + f\theta_1' - 2\beta f'\theta_1 = \theta_0'', \tag{2.8}$$

$$\theta_2'' + f\theta_2' - 2\beta f'\theta_2 = \theta_1'' - \theta_0'', \tag{2.9}$$

or, in general (for $i > 1$)

$$\theta_i'' + f\theta_i' - 2\beta f'\theta_i = G_i \tag{2.10}$$

where $G_i = \theta_{i-1}'' - G_{i-1}$. For each function the same boundary conditions $\theta_i'(0) = 0$ and $\theta_i(\infty) = 0$ apply. From (2.6) the recovery factor $r$ follows as

$$r = \theta(0) = \theta_0(0) + \epsilon\theta_1(0) + \epsilon^2\theta_2(0) + \ldots = c_0(m) + \epsilon c_1(m) + \epsilon^2 c_2(m) + \ldots \tag{2.11}$$

with the coefficients $c_i$ in general depending on the pressure-gradient parameter $m$.

Since $\theta(\eta)$ represents an exact solution of the energy equation, it naturally satisfies the integral condition (1.6). The same therefore holds for the asymptotic series of (2.6) for any $\epsilon$ for which it converges. As it does so for arbitrary values of $Pr$, it can be concluded that any truncation of the series satisfies the integral condition as well.

The zero-order solution is identical to the solution of the original problem with $Pr = 1$, so that in correspondence with the original Crocco integral relation

$$\theta_0 = 1 - f'^2 \tag{2.12}$$

which, upon substitution in (2.8) yields the first-order problem as

$$\theta_1'' + f\theta_1' - 2\beta f'\theta_1 = -2(f''^2 + f'f''') = -2(f'f'')'. \tag{2.13}$$

For the flat plate ($\beta = 0$) the problem is particularly easy to solve, as in that case $\theta_1 = f'$ is directly recognized as a solution of the homogeneous equation (as it is for every $\theta_i$), allowing the solution to the non-homogeneous equation to be obtained by the method of variation of parameters. The solution for $\theta_1$ is found as

$$\theta_1 = \tfrac{1}{2}(1 - f'^2) - ff''. \tag{2.14}$$

Substitution of this expression in (2.13) reveals that, remarkably, this satisfies for general $\beta$ as well. The second-order solution can also be derived explicitly for the flat-plate case, but does not possess the same universal validity for arbitrary $\beta$ as does the first-order solution.

| | $f''(0)$ | $\theta(0)$ $(Pr = 0.7)$ | $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|---|---|---|
| $m = -0.0904$ | 0.0071 | 0.8359 | 1.0000 | 0.5000 | −0.1331 | 0.0641 | −0.0389 |
| $m = -0.075$ | 0.1849 | 0.8358 | 1.0000 | 0.5000 | −0.1338 | 0.0646 | −0.0392 |
| $m = -0.05$ | 0.3098 | 0.8358 | 1.0000 | 0.5000 | −0.1342 | 0.0651 | −0.0396 |
| $m = 0$ | 0.4696 | 0.8357 | 1.0000 | 0.5000 | −0.1345 | 0.0655 | −0.0399 |
| $m = 0.5$ | 1.0389 | 0.8359 | 1.0000 | 0.5000 | −0.1348 | 0.0661 | −0.0404 |
| $m = 1$ | 1.2326 | 0.8360 | 1.0000 | 0.5000 | −0.1349 | 0.0661 | −0.0404 |
| $r = Pr^{1/2}$ | | 0.8367 | 1 | 1/2 | −1/8 | 1/16 | −0.0391 |
| Spence (1960) | | 0.8357 | 1 | 1/2 | −0.1345 | | |
| Le Fur (1960) | | 0.835 | 1 | 1/2 | −1/7 | 1/14 | |

TABLE 1. Enthalpy recovery in constant-property similar boundary layer flow. Numerical results are obtained with step size $\Delta\eta = 0.02$ and $\eta_{max} = 10$.

A short numerical investigation was made, with the purpose of obtaining solutions to the complete $\theta$-equation for a given value of $Pr$, as well as for evaluating the different functions $\theta_i(\eta)$ that feature in the asymptotic approach. A five-point equidistant finite-difference scheme with fourth-order accuracy was employed, that solves a second-order linear differential equation by a modified version of the Thomas algorithm (cf. Schetz 1993); the nonlinear $f$-equation is solved iteratively by subsequent approximations. The boundary conditions pertaining to the external stream are applied at a finite value $\eta_{max}$ that is taken sufficiently large so as to ensure that the solution does not depend on it within numerical accuracy.

Results for different values of the pressure-gradient parameter $m$ are given in table 1. Note that in the light of the previous analytical findings the first two coefficients in (2.11) are independent of $m$, with $c_0 = 1$ and $c_1 = \frac{1}{2}$. The subsequent coefficients show only a very weak dependence on $m$. Also included in the table are the coefficients of the expansion of the common approximation $r = Pr^{1/2}$, the second term for the flat plate as derived by Spence (1960) which is seen to agree with the present data for $m = 0$, as well as the coefficients obtained by Le Fur (1960).

Profiles of the scaled distributions of velocity, enthalpy and total enthalpy are shown in figure 1, for flat-plate, stagnation and separation flow ($m = 0$, $m = 1$ and $m = -0.0904$, respectively) with $Pr = 0.7$. In the figure the numerical results of the enthalpy distribution are compared to the predictions of the Crocco integral with either the classic modification of (1.3), or with the present result as given by (3.3). To improve correspondence with the numerical results, in both Crocco integral methods the recovery factor is taken as $r = Pr^{1/2}$, as a representation of the higher-order Prandtl-number effects on $r$.

Clearly the improved prediction that the extended Crocco integral provides of the static and total enthalpy in comparison to the standard modification can be seen, in particular the overshoot of $H$ in the outer region of the boundary layer is correctly modelled.

## 3. Generalization of the results

### 3.1. *Interpretation in dimensional properties*

Returning to dimensional properties, and with substitution of the results that were obtained for the first two terms $\theta_0$ and $\theta_1$ of (2.6), the Crocco integral and recovery

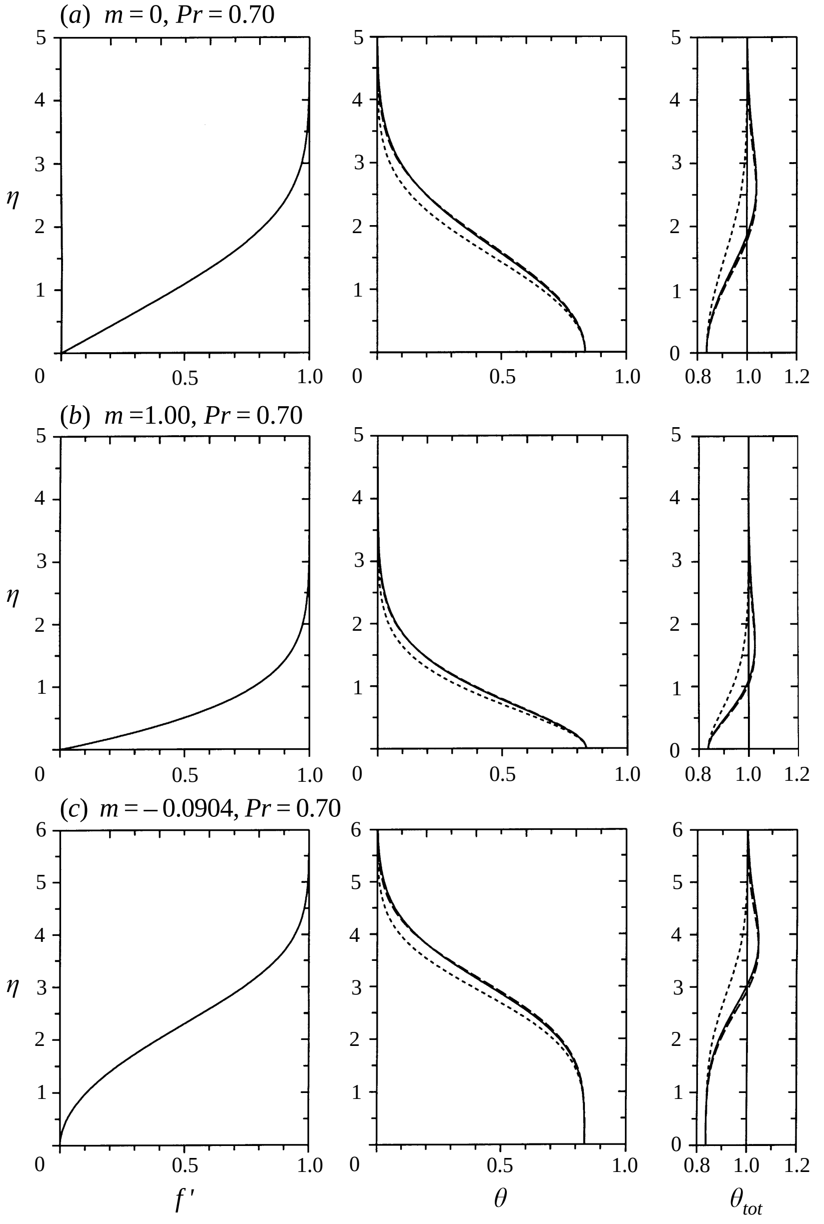


FIGURE 1. Boundary layer profiles of velocity, enthalpy and total enthalpy ($\theta_{tot} = \theta + f'^2$) for constant-property self-similar flow. (*a*) Flat-plate flow; (*b*) stagnation flow; (*c*) separation flow. Solid line: numerical solution of the energy equation; short-dashed line: classic Crocco integral, equation (1.3); long-dashed line: extended Crocco integral, equation (3.3).

factor are expressed explicit up to first order in $\epsilon$ as:

$$h = h_e + \tfrac{1}{2}(u_e^2 - u^2) + \tfrac{1}{2}\epsilon\left(\tfrac{1}{2}(u_e^2 - u^2) - \psi\frac{\partial u}{\partial y}\right) + O(\epsilon^2), \tag{3.1}$$

$$r = 1 + \tfrac{1}{2}\epsilon + O(\epsilon^2), \tag{3.2}$$

where $\psi$ is the stream function defined by $u = \partial\psi/\partial y$. After truncation of both series by neglecting the $O(\epsilon^2)$ terms and with subsequent elimination of $\epsilon$, the relation between the enthalpy and the velocity can be written alternatively as

$$h = h_e + r\tfrac{1}{2}(u_e^2 - u^2) - (r-1)\psi\frac{\partial u}{\partial y} \tag{3.3}$$

or

$$H = H_e + (r-1)\left(\tfrac{1}{2}(u_e^2 - u^2) - \psi\frac{\partial u}{\partial y}\right). \tag{3.4}$$

Comparison with the classic forms of the modified Crocco relation, see (1.3) and (1.4), reveals that the latter incorporate only the first recovery part of the first-order term, but that the second part that expresses the energy migration towards the outer flow is absent in them.

As mentioned previously, being a truncated series of the exact solution, (3.4) inherently satisfies the integral energy-conservation requirement of (1.6). It does so, not only for the similar boundary layer flows for which it was derived, but also in general, as can be confirmed from substitution of (3.4) in (1.6). Noting that $\int \rho u^3 \mathrm{d}y = \int \rho u^2 \mathrm{d}\psi$, the following result is obtained by means of partial integration:

$$\int_0 \rho u(H - H_e)\mathrm{d}y = \tfrac{1}{2}(r-1)\rho\psi(u_e^2 - u^2), \tag{3.5}$$

which indeed vanishes when the upper bound of the integration extends beyond the boundary layer edge.

For the numerical solutions shown in figure 1 the Crocco relation is plotted in figure 2 in the form of the scaled total-enthalpy defect,

$$\frac{H - h_w}{H_e - h_w} = \frac{\theta + f'^2 - \theta(0)}{1 - \theta(0)}, \tag{3.6}$$

versus $(u/u_e)^2 = f'^2$, as suggested by the alternative form of (1.1) (Walz 1966; Bushnell *et al.* 1969; Fernholz & Finley 1980):

$$\frac{H - h_w}{H_e - h_w} = \frac{H_e - h_{aw}}{H_e - h_w}\left(\frac{u^2}{u_e^2}\right) + \frac{h_{aw} - h_w}{H_e - h_w}\left(\frac{u}{u_e}\right). \tag{3.7}$$

For an adiabatic wall the second term vanishes, so that accordingly the common modification of the Crocco relation predicts a quadratic dependence on the velocity and, hence, a linear relation between the variables of the figure, which is clearly not in agreement here with the solutions of the similarity equations, cf. also Whitfield & High (1977). Instead, with the present results of (3.4), we find

$$\frac{H - h_w}{H_e - h_w} = \frac{u^2}{u_e^2} + \frac{\psi}{u_e^2}\frac{\partial u}{\partial y}. \tag{3.8}$$

### 3.2. *General constant-property flow*

The observations in the previous subsection, together with the result that (3.3) is valid for self-similar boundary layers irrespective of the value of the pressure-gradient parameter $m$, suggest that it may possess an even more general validity. It can indeed be verified that (3.1) represents a general solution to the energy equation, (2.2), in the same asymptotic sense as considered previously for the self-similar boundary layers.

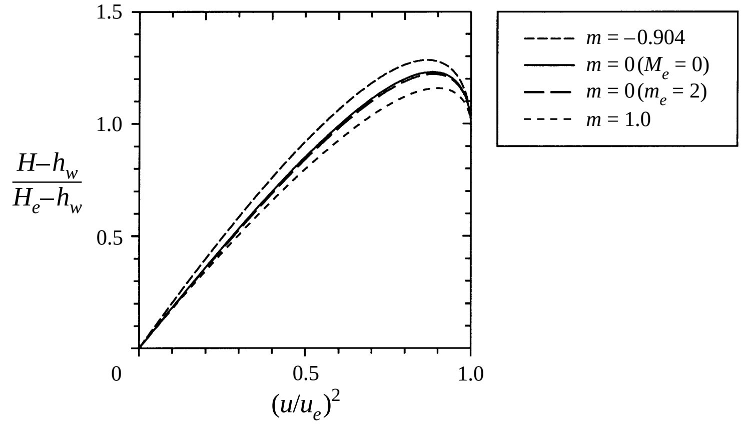


FIGURE 2. Representation of total enthalpy defect in 'Crocco variables' (data are for $Pr = 0.7$ and $\omega = 0.75$).

To show this, the asymptotic approach is applied directly to the enthalpy distribution:

$$h(x, y) = h_0(x, y) + \epsilon h_1(x, y) + \epsilon^2 h_2(x, y) + \ldots \tag{3.9}$$

from which the zero-order problem is found to be

$$\rho u \frac{\partial h_0}{\partial x} + \rho v \frac{\partial h_0}{\partial y} - \frac{\partial}{\partial y} \mu \frac{\partial h_0}{\partial y} = u \frac{\mathrm{d}p}{\mathrm{d}x} + \mu \left( \frac{\partial u}{\partial y} \right)^2 \tag{3.10}$$

with solution $h_0 = h_e + \frac{1}{2}(u_e^2 - u^2)$, as dictated by the universal validity of the Crocco integral for $Pr = 1$. Consequently, the first-order problem is described by

$$\rho u \frac{\partial h_1}{\partial x} + \rho v \frac{\partial h_1}{\partial y} - \frac{\partial}{\partial y} \mu \frac{\partial h_1}{\partial y} = - \frac{\partial}{\partial y} \mu \frac{\partial h_0}{\partial y} = \frac{\partial}{\partial y} \mu u \frac{\partial u}{\partial y}. \tag{3.11}$$

To verify that, in accordance with (3.1), $h_1 = \frac{1}{4}(u_e^2 - u^2) - \frac{1}{2}\psi \partial u / \partial y$ satisfies as a solution, the following operator is evaluated:

$$\mathscr{F}(h_1) = \rho u \frac{\partial h_1}{\partial x} + \rho v \frac{\partial h_1}{\partial y} - \frac{\partial}{\partial y} \mu \frac{\partial h_1}{\partial y} - \frac{\partial}{\partial y} \mu u \frac{\partial u}{\partial y}. \tag{3.12}$$

After substitution of the expression for $h_1$ given above, and with some manipulation, the following identity is derived:

$$\mathscr{F}(h_1) = -\frac{1}{2} \left( \rho u \frac{\partial}{\partial y} \psi \frac{\partial u}{\partial x} + \rho v \frac{\partial}{\partial y} \psi \frac{\partial u}{\partial y} - \rho u u_e \frac{\mathrm{d}u_e}{\mathrm{d}x} - \frac{\partial}{\partial y} \mu \psi \frac{\partial^2 u}{\partial y^2} \right), \tag{3.13}$$

which, invoking the continuity equation and noting that $u = \partial \psi / \partial y$, can be evaluated as

$$\mathscr{F}(h_1) = -\frac{1}{2} \frac{\partial}{\partial y} \psi \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \rho u_e \frac{\mathrm{d}u_e}{\mathrm{d}x} - \mu \frac{\partial^2 u}{\partial y^2} \right) \equiv 0 \tag{3.14}$$

which indeed vanishes as follows from the momentum equation (2.1), provided that $\rho$ and $\mu$ are invariant with $y$, which proves the correctness of the assumed solution for $h_1$.

## 4. Compressible flow

In the case of varying properties that occur in compressible flow, the possibility of obtaining strict similarity is very limited (Li & Nagamatsu 1955; Stewartson 1964). The reason for this is that, in addition to similarity in the enthalpy-defect function $\theta$, the dependence of density and viscosity on temperature requires similarity in the enthalpy $h$ itself as well (Anderson 1989). With the relation between $h$ and $\theta$ being given by

$$\frac{h}{h_e} = 1 + \frac{\gamma - 1}{2} M_e^2 \theta(\eta) \tag{4.1}$$

this directly reveals that strict similarity in general can only be obtained when $M_e$ is constant (flat-plate flow), or when compressibility effects are negligible, the latter basically reducing the problem to that of constant-property flow which has been addressed in the previous section. Alternatively, solutions with approximate similarity can be obtained under certain restrictions (Li & Nagamatsu 1955), for heat-transfer-dominated boundary layers (Chapman & Rubesin 1949) or assuming only local similarity (Anderson 1989).

For mild compressibility effects the asymptotic approach can be extended by introducing a second perturbation parameter $\epsilon_M = \frac{1}{2}(\gamma - 1)M_e^2$ that expresses the compressibility effect. The asymptotic series for $h$, equation (3.9), is then written

$$h(x, y) = h_{00}(x, y) + \epsilon_M h_{01}(x, y) + \ldots + \epsilon\,(h_{10}(x, y) + \epsilon_M h_{11}(x, y) + \ldots) + \ldots. \tag{4.2}$$

As a result of the coupling of momentum and energy equations due to the varying properties, similar series expansions must now also be considered for the various velocity components, like

$$u(x, y) = u_{00}(x, y) + \epsilon_M u_{01}(x, y) + \ldots + \epsilon\,(u_{10}(x, y) + \epsilon_M u_{11}(x, y) + \ldots) + \ldots. \tag{4.3}$$

Assuming constant specific heat, the variation of the properties $\mu$ and $\rho$ are directly coupled to the enthalpy by, respectively, the viscosity law and the equation of state, with in the boundary layer the latter reducing to $\rho/\rho_e = h_e/h$.

The (00)-problem yields the basic solution for $Pr = 1$ and $M = 0$, while the (01) and the (10) cases describe the first-order corrections for variations in $M$ and $Pr$, respectively. As the original Crocco integral for $Pr = 1$ is valid for variable properties as well, it can directly be concluded that (4.2) with $\epsilon = 0$ then reads $h_0 = h_e + \frac{1}{2}(u_e^2 - u_0^2)$, hence:

$$h_{00} = h_e + \tfrac{1}{2}(u_e^2 - u_{00}^2), \qquad h_{01} = -u_{00}u_{01}. \tag{4.4}$$

Furthermore, the (10)-problem describes the effect of Prandtl number in the absence of compressibility effects ($\epsilon_M = 0$), so the results of the previous section are directly applicable here. In particular, we find that

$$u_{10} \equiv 0, \qquad h_{10} = \tfrac{1}{2}\left(\tfrac{1}{2}(u_e^2 - u_{00}^2) - \frac{\Psi_{00}}{\rho_{00}}\frac{\partial u_{00}}{\partial y}\right), \tag{4.5}$$

where the extra factor density has been included in view of the adapted definition of the stream function $\Psi$ in compressible flow, being $\rho u = \partial\Psi/\partial y$. Note that this modification is also in agreement with the common feature of all compressibility transformations used to reduce the flow equations to a nearly incompressible form (Anderson 1989; White 1991; Schetz 1993), that a density-weighing is applied to the transverse coordinate, hence directly suggesting that $\partial y$ should be replaced by $\rho\partial y$.

As a result, all relations established in §3.1 apply here as well with error $O(\epsilon\epsilon_M)$, that is, incorporating only first-order $Pr$ and $M$ effects separately. In particular the generalization of the Crocco integral applies:

$$h = h_e + r\tfrac{1}{2}(u_e^2 - u^2) - (r-1)\frac{\Psi}{\rho}\frac{\partial u}{\partial y} + O(\epsilon^2, \epsilon\epsilon_M, \epsilon_M^2), \tag{4.6}$$

$$r = 1 + \tfrac{1}{2}\epsilon + O(\epsilon^2, \epsilon\epsilon_M, \epsilon_M^2). \tag{4.7}$$

### 4.1. *Compressible flat-plate flow*

In order to investigate the first-order combined effect described by the (11)-term, the case of flat-plate flow is considered in more detail. As in this case a similarity solution can be obtained, it is attractive for analytical treatment, and has received this extensively. Special reference can be made to the work by Van Driest (1952, 1959), who numerically investigated the effect of Mach number and viscosity law on velocity and temperature distributions, skin friction, heat transfer and recovery factor, while including in the analysis the temperature dependence of specific heat and Prandtl number. In particular, he showed that the recovery factor is not affected by compressibility effects for a linear viscosity law $\mu \propto T$, but in general $r$ is a function of both $M_e$ and $T_e$.

Application of a standard compressibility transformation by including the free-stream properties $\mu_e$, $\rho_e$ and $u_e$ in the coordinate scaling, which becomes especially simple for the flat-plate flow where these properties are constants, brings the similarity equations into a nearly incompressible form (an extensive derivation is given in Anderson 1989):

$$(Cf'')' + ff'' = 0, \tag{4.8}$$

$$(C\theta')' + Prf\theta' = -2PrCf''^2, \tag{4.9}$$

which is identical to the incompressible form apart from the Chapman–Rubesin function $C = \rho\mu/\rho_e\mu_e$ (Chapman & Rubesin 1949). For a general viscosity law the solution is a function of both $M_e$ and $T_e$, but when the viscosity is expressed as a power law $\mu/\mu_e = (T/T_e)^\omega$, the effect of $T_e$ vanishes. The power-law function will be employed here for convenience, and for the present purpose of investigating only mild compressibility effects, it suffices in providing an approximate description of the variation of $\mu$ over a limited temperature range. Under that assumption we obtain for $C$

$$C = (h/h_e)^{\omega-1} = (1 + \epsilon_M\theta(\eta))^{\omega-1}, \tag{4.10}$$

which for weak compressibility effects and/or $\omega$ near 1, can be approximated as

$$C = 1 + (\omega - 1)\epsilon_M\theta(\eta). \tag{4.11}$$

This shows that under these conditions an effective perturbation parameter for the effect of compressibility can be defined as $\epsilon_m = (\omega - 1)\epsilon_M$. When $\omega = 1$ the compressibility effects vanish completely from the problem as $\epsilon_m \equiv 0$, which is obvious, as $C \equiv 1$ makes the governing equations identical to their incompressible counterparts. Hence, under these conditions the results of the constant-property flow as expressed for example by (3.1) and (3.2) are directly applicable as well.

In figure 3 the results of calculations with $\omega = 0.75$ for free-stream Mach numbers $M_e$ of 1 and 2 are shown. The effect of $M_e$ on the validity of the Crocco relation appears to be very small, as is further revealed by comparison with the constant-property calculations, see also figure 1(*a*) and figure 2.

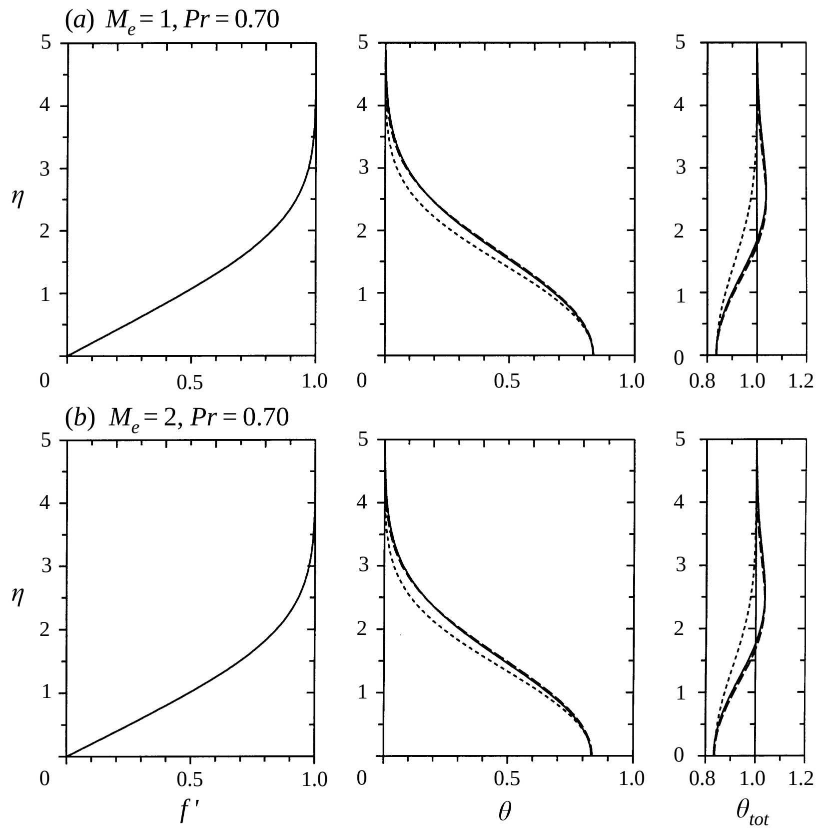


FIGURE 3. Boundary layer profiles for compressible flat-plate flow. (*a*) $M_e = 1$; (*b*) $M_e = 2$. For line styles see figure 1 ($\omega = 0.75$).

For the case $\omega \neq 1$ asymptotic series in both $\epsilon$ are $\epsilon_m$ are constructed for $f$ and $\theta$ as

$$f(\eta) = f_{00}(\eta) + \epsilon_m f_{01}(\eta) + \ldots + \epsilon\,(f_{10}(\eta) + \epsilon_m f_{11}(\eta) + \ldots) + \ldots, \tag{4.12}$$

$$\theta(\eta) = \theta_{00}(\eta) + \epsilon_m \theta_{01}(\eta) + \ldots + \epsilon\,(\theta_{10}(\eta) + \epsilon_m \theta_{11}(\eta) + \ldots) + \ldots. \tag{4.13}$$

From the discussion in the previous section the solutions for $\theta_{00}$ and $\theta_{01}$ can directly be expressed as follows:

$$\theta_{00} = 1 - f_{00}'^2, \qquad \theta_{01} = -\tfrac{1}{2} f_{00}' f_{01}', \tag{4.14}$$

where $f_{00}$ satisfies the Blasius equation, $f_{00}''' + f_{00} f_{00}'' = 0$, while $f_{01}$, the function expressing the first-order compressibility effect on the velocity profile, is determined by the following linear differential equation:

$$f_{01}''' + f_{00} f_{01}'' + f_{00}'' f_{01} + (\theta_{00} f_{00}'')' = 0. \tag{4.15}$$

Also, the solutions for $f_{10}$ and $\theta_{10}$ are directly obvious from the constant-property analysis, yielding

$$f_{10} \equiv 0, \qquad \theta_{10} = \tfrac{1}{2}(1 - f_{00}'^2) - f_{00} f_{00}''. \tag{4.16}$$

while the (11)-problem is described by

$$f_{11}''' + f_{00} f_{11}'' + f_{00}'' f_{11} + (\theta_{10} f_{00}'')' = 0, \tag{4.17}$$

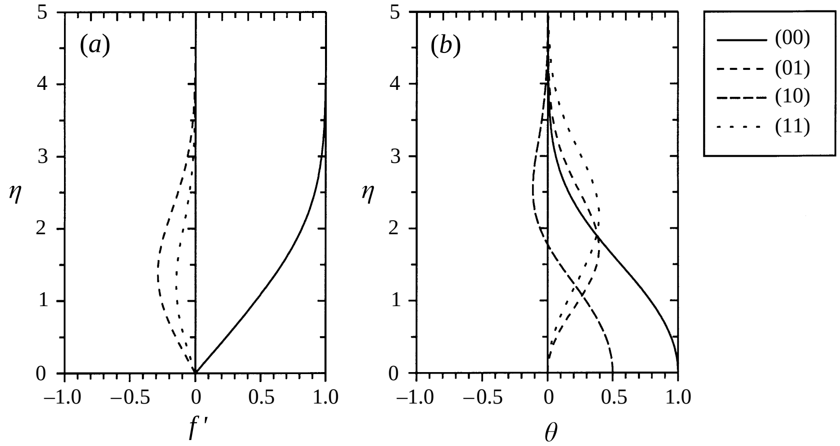


FIGURE 4. Asymptotic functions for the velocity and enthalpy distribution in compressible flat-plate flow.

$$\theta''_{11} + f_{00}\theta'_{11} + (\theta_{10}\theta'_{00})' + (\theta_{00}\theta'_{10})' + (f_{01} + f_{11})\theta'_{00} + f_{01}\theta'_{10} + f_{00}\theta'_{01} \\ +2(\theta_{00} + \theta_{10})f''^2_{00} + 4(f''_{01} + f''_{11})f''_{00} = 0. \quad (4.18)$$

The numerically obtained asymptotic functions for $f'$ and $\theta$, applying to the series expansions of the velocity and enthalpy distribution, have been depicted in figure 4. Table 2 contains the relevant values of $f''(0)$ and $\theta(0)$, which relate to the wall shear stress and recovery factor, as well as results from the numerical solution for several values of the external stream Mach number $M_e$, for $Pr = 1$ and $Pr = 0.7$ (calculations are again for $\omega = 0.75$). Figure 5 provides a comparison between the numerical results and the predicted effect of $M_e$ on $f''(0)$ and $r = \theta(0)$, according to the asymptotic solution

$$f''(0) = f''_{00}(0) + \epsilon_m \left(f''_{01}(0) + \epsilon f''_{11}(0)\right), \quad (4.19)$$

$$r = r_0 + \epsilon_m \epsilon \theta_{11}(0), \quad (4.20)$$

where in the latter, instead of taking only the first-order expansion, $r_0$ is fitted to the result of the numerical solution for $M_e = 0$, in order to observe the $M_e$-effect separately from the influence of $Pr$. Note especially the small variation of $r$ with $M_e$ in the prediction, and which is even smaller when the higher-order terms are included as the numerical solutions show. This result is also in good agreement with the predicted compressibility effect on $r$ as derived by Herwig (1987) from a perturbation of the constant-property solution for $Pr = 0.7$, see also Schlichting & Gersten (1997), which yields for $\gamma = 1.4$

$$r = 0.836 - 0.003(1 - \omega)M_e^2, \quad (4.21)$$

whereas the present results correspond to

$$r = r_0 - 0.004(1 - \omega)M_e^2. \quad (4.22)$$

Apart from numerical accuracy, the differences are to be attributed to the fact that the second expression is obtained from the perturbation of the constant-property solution for $Pr = 1$, instead of $Pr = 0.7$.

The above results mainly serve to illustrate that the first-order effect of compressibility on the constant-property analysis is not very large, at least not for the flat-plate

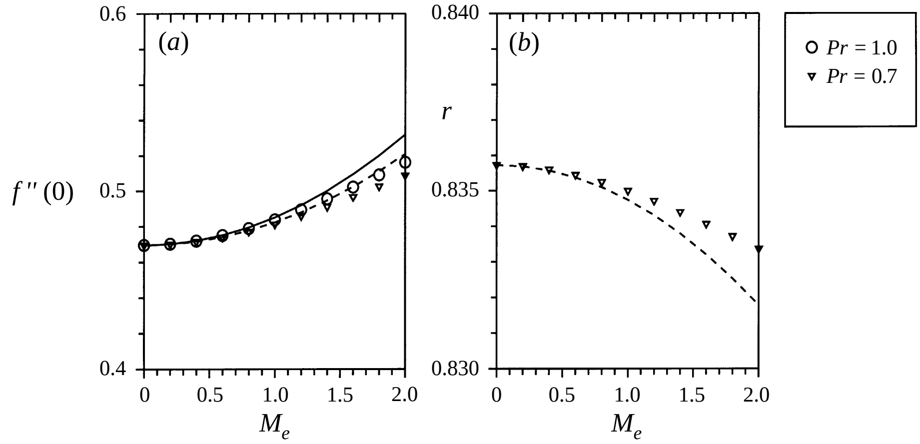


FIGURE 5. Mach number effect on the scaled wall shear stress and recovery factor for flat-plate flow. Symbols: numerical solution of the energy equation; solid line: prediction for $Pr = 1$; dashed line: prediction for $Pr = 0.7 (\omega = 0.75)$.

(*a*) Properties of asymptotic functions.

| | (00) | (01) | (10) | (11) |
|---|---|---|---|---|
| $f''(0)$ | 0.4696 | −0.3113 | 0 | 0.1838 |
| $r = \theta(0)$ | 1 | 0 | 0.5 | −0.0657 |

(*b*) Solutions for the original system of equations.

| | | $M_e = 0$ | $M_e = 1$ | $M_e = 2$ | $M_e = 3$ | $M_e = 4$ |
|---|---|---|---|---|---|---|
| $f''(0)$ | $Pr = 0.7$ | 0.4696 | 0.4813 | 0.5087 | 0.5404 | 0.5710 |
| | $Pr = 1$ | 0.4696 | 0.4839 | 0.5161 | 0.5522 | 0.5861 |
| $\theta(0)$ | $Pr = 0.7$ | 0.8357 | 0.8350 | 0.8334 | 0.8316 | 0.8301 |
| | $Pr = 1$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

TABLE 2. Calculation results for compressible flat-plate boundary layer flow ($\omega = 0.75$).

flow. More detailed consideration of compressibility effects appears to be meaningful only if the temperature dependence of specific heat and Prandtl number are taken into account as well (cf. Van Driest 1959).

## 5. Some remarks on three-dimensional flow effects

The increased complexity of heat transfer in three-dimensional flow prevents a rigourous analytical approach of this matter in general, apart from degenerate three-dimensionality such as occurs in axisymmetric and infinite swept geometries (Stojanovic 1959; Reshotko & Beckwith 1958). Possibly the only significant analytical landmark to be claimed here is the fact that the Crocco integral may be readily extended to three-dimensional boundary layers, under the same conditions of unity Prandtl number and zero surface heat and mass transfer, but allowing variable properties, to read (White 1991)

$$h = h_e + \tfrac{1}{2}(q_e^2 - q^2) \tag{5.1}$$

where $q^2 = u^2 + w^2$ is representative of the total main velocity vector, with $w$ the additional velocity component in the spanwise $z$-direction.

As far as the effect of Prandtl number is concerned, no attempt is known to the author of any strict proof that the expression for the recovery factor $r = Pr^{1/2}$ should be applicable to three-dimensional flow as well, not even in the case of constant-property flow. On the contrary, the strict validity of this dependence in the asymptotic sense that was found for two-dimensional flow is not present in the case of three-dimensional flow, as illustrated by the example for yawed wedge flow that will be considered below. This generalization, therefore, appears to be no more than a fortuitous approximate result, supported only by discrete data of numerical or experimental investigations of particular three-dimensional flow fields.

An *ad hoc* modification of the Crocco integral for non-unity $Pr$ in three-dimensional flow commonly takes the form of including a recovery factor $r$ in (5.1) in the same form as expressed by (1.3), i.e. replacing $u$ in the expressions for two-dimensional flow by $q$ in three-dimensional flow.

No logical extension of the 'complete' Crocco integral introduced in the present study appears to be possible. Although the same method of replacing $u$ by $q$ may be suggested, there is no fundamental justification for it. The underlying physical explanation is that for three-dimensional conditions a local integral energy requirement of the type of equation (1.6) is absent in general. Instead, the energy equation for an adiabatic wall integrates to

$$\frac{\partial}{\partial x}\int_0^{\delta} \rho u(H - H_e)\mathrm{d}y + \frac{\partial}{\partial z}\int_0^{\delta} \rho w(H - H_e)\mathrm{d}y = 0. \tag{5.2}$$

This can be considered as a transport expression of the total amount of enthalpy defect present in the boundary layer, in accordance with the parabolic nature of the boundary layer equations. In two-dimensional flow the total boundary layer enthalpy defect is propagated only in a single dimension, hence in the absence of heat transfer its value must be the same at every $x$-position, as expressed by (1.6). In three-dimensional flow this is no longer the case, as enthalpy can be distributed by propagation in two dimensions. Only when $Pr = 1$ are the enthalpy transport by thermal diffusion and viscous work locally in balance, everywhere in the flow. As a result the total enthalpy is convected with the flow, and hence remains constant along streamlines, as is expressed by the original Crocco relation being valid for variable-property three-dimensional flow in general.

### 5.1. *Yawed constant-property wedge flow*

As an example of three-dimensional flow, albeit with a degenerate three-dimensionality, let the similarity solution be considered that applies to yawed infinite wedge flow. These flows are obtained by adding to the external flow a homogeneous flow of constant velocity $w_e$ in the spanwise $z$-direction, whereas $x$ is measured perpendicular to the leading edge. In this respect especially the yawed stagnation flow has significant practical relevance to the flow near the attachment line of swept wings.

The velocity problem was treated by Cooke (1950) for incompressible flow. For compressible flow Crabtree (1954) considered the solution for $Pr = 1$, while the case of arbitrary $Pr$ was reported by Reshotko & Beckwith (1958).

For constant-property flow the governing transformed equations for $x$- and $z$-momentum and for energy are obtained under the same transformation as for the

| | $f''(0)$ | $g'(0)$ | $\theta(0)$ $(Pr = 0.7)$ | $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|---|---|---|---|
| $m = 0$ | 0.4696 | 0.4696 | 0.8357 | 1.0000 | 0.5000 | −0.1345 | 0.0655 | −0.0399 |
| $m = 0.1$ | 0.6696 | 0.5042 | 0.8403 | 1.0000 | 0.4847 | −0.1342 | 0.0664 | −0.0409 |
| $m = 0.25$ | 0.8544 | 0.5300 | 0.8436 | 1.0000 | 0.4739 | −0.1339 | 0.0670 | −0.0415 |
| $m = 0.5$ | 1.0389 | 0.5515 | 0.8462 | 1.0000 | 0.4651 | −0.1335 | 0.0674 | −0.0421 |
| $m = 0.75$ | 1.1534 | 0.5631 | 0.8476 | 1.0000 | 0.4604 | −0.1333 | 0.0677 | −0.0424 |
| $m = 1$ | 1.2326 | 0.5705 | 0.8485 | 1.0000 | 0.4573 | −0.1332 | 0.0678 | −0.0426 |

TABLE 3. Calculation results for constant-property yawed infinite wedge flow.

non-swept wedge flow, yielding

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{5.3}$$

$$g'' + fg' = 0, \tag{5.4}$$

$$\theta'' + Prf\theta' = -2Prg'^2, \tag{5.5}$$

where $f' = u/u_e$, $g = w/w_e$ and $\theta = 2(h - h_e)/q_e^2$. The additional boundary conditions applying to $g$ are $g(0) = 0$ and $g(\infty) = 1$. The similarity in the energy equation has been achieved by neglecting $u$ with respect to $w$ in the kinetic energy, i.e. restricting the problem to the vicinity of the attachment line, so that $q_e \approx w_e$. As this requires $u_e$ to vanish near the attachment line, the analysis applies to $\beta > 0$. However, this restriction need not be made for the flat-plate case $\beta = 0$, where with $g \equiv f'$ direct similarity is obtained for $q_e^2 = u_e^2 + w_e^2$. The analysis may therefore be considered valid for $\beta \geqslant 0$.

Applying a similar asymptotic solution method as in the non-swept case, by writing $\theta(\eta)$ in the form of (2.6), the following result is obtained for the subsequent terms of the asymptotic expansion:

$$\theta_0'' + f\theta_0' + 2g'^2 = 0, \tag{5.6}$$

$$\theta_1'' + f\theta_1' = \theta_0'', \tag{5.7}$$

or, in general (for $i > 1$)

$$\theta_i'' + f\theta_i' = G_i \tag{5.8}$$

with $G_i = \theta_{i-1}'' - G_{i-1}$. Results of numerical solutions for different values of the pressure-gradient parameter $m$ have been collected in table 3. The pressure gradient is seen to have a notable effect on $c_1$, with the value for the stagnation-line solution ($m = 1$) in agreement with the approximation $r = Pr^{0.46}$ obtained by Reshotko & Beckwith (1958).

From the above results it becomes obvious that the approximation $r = Pr^{1/2}$ does not possess the same universal validity in three-dimensional flow as it does in two-dimensional flow. Note that even the degenerate three-dimensionality of infinite swept flow, where as a result of the vanishing of the $z$-derivatives the integral energy relation of (5.2) is reduced to the two-dimensional form of (1.6), does not reproduce this relation.

## 6. Conclusions

The extension of the Crocco integral for Prandtl numbers different from unity was investigated. A fundamental objection to the common extension, obtained by applying a constant recovery factor in the entire boundary layer, was pointed out, namely that it violates the conservation-of-energy principle.

Based on an investigation of self-similar boundary layers in two-dimensional constant-property flow, a relation was established between the enthalpy and velocity fields, that provides the correct first-order asymptotic description of the $Pr$ effect. In addition to the 'recovery term' it contains a second term resulting from the redistribution of energy over the boundary layer. Being an asymptotically correct truncation of the full solution, it is complete in the sense that it satisfies the integral energy requirement dictated by conservation principles. The result can be generalized in that it provides a valid first-order solution for constant-property flow in general and in the presence of weak compressibility effects.

This expression is evidently more complex than the simple classic modified Crocco integral, where the enthalpy is related directly to the local flow velocity. However, such a direct relation cannot be expected to possess strict validity, unless heat conduction and viscous work are locally in balance, as is the case only for $Pr = 1$. The extended Crocco integral derived in the present study expresses the local enthalpy in terms of the complete velocity profile at a given location. The underlying physical principle for this is the one-dimensional propagation of the 'total boundary layer enthalpy defect' that occurs in two-dimensional flow. This directly suggests that an extension of this result to three-dimensional boundary layers with the same general validity, i.e. relating the local enthalpy and velocity profiles, is not possible except for the trivial case $Pr = 1$.

In addition to the asymptotic expression for the Crocco integral, asymptotic solutions have also been obtained for the recovery factor for various classes of flows. This shows that the expression $r = Pr^{1/2}$ is asymptotically correct to first order for two-dimensional constant-property flow in general, and approximately so for weak compressibility effects. For three-dimensional flow it can only be approximate, even for constant-property flow, as shown by the example of the class of swept wedge flows, where a distinct pressure-gradient effect on $r$ was observed.

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